Solutions for Go Figure 2001

1. 16%. Expressed as a decimal, 20% = .2 and 30% = .3. Therefore

$$\begin{aligned} B &= 100 + .2 \times 100 \\ &= 100 + 20 \\ &= 120, \end{aligned} \qquad \begin{aligned} C &= 120 - .3 \times 120 \\ &= 120 - 36 \\ &= 84. \end{aligned}$$

   $A - C = 100 - 84 = 16$, so A must be reduced by 16% to get C.

2. $m = 60, a = 12, b = 10$. $m = 30 \div \frac{1}{2} = 30 \times 2 = 60, a = 60 \times \frac{1}{5} = 12, b = 60 \times \frac{1}{6} = 10$.

3. (a) 276. In this progression, the terms are $3 \times 1, 3 \times 2, 3 \times 3, 3 \times 4 \ldots$ The 92nd term is $3 \times 92 = 276$.

   (b) 275. Each term in the sequence in part (b) is one less that the corresponding term in the sequence in part (a). (b) The 92nd term in (a) is 276, so the 92nd term in (b) is $276 - 1 = 275$.

4. 231. The sequence is an arithmetic progression with a fixed difference of 2. If this sequence had begun with 2, we would only have to divide the final term by 2 to find the number of terms. But as we have seen in Problem 3, we can compare sequences with the same fixed difference and length. Therefore, the sequence $2, 4, 6, 8 \ldots, 462$ (where each term is 5 less that the corresponding term in our sequence) has the the same number of terms as our sequence. So the number of terms is $462/2 = 231$.

   A "classic" method to determine the number of sequence elements between a start value and and end value is to compute the span of the sequence (difference between the first and last term): $467 - 7 = 460$. Dividing by the difference between successive terms (in this case 2) gives us the number of terms *after the first one*: $460/2 = 230$ terms after the first one. We must then add one for the first term, so there are 231 in total. You should use this method with caution since it is easy to forget to add one for the first term.

5. $A = 9, B = 5, C = 2$. The product has six digits, so A must be large (or the product would have only five digits). Since 104×980 is just over $100,000$, A must be 9. We have $B = 5$ because $5 \times 4 = 20$ is the only product of 4 with a single-digit number that has a 0 in the units digit. Therefore the second number is 985. Computing the product, $104 \times 985 = 102440$, which is of the correct form with $C = 2$.

6. (a) 6. There are four groups of numbers in S, each group containing the numbers with a given digit in the thousands place (numbers with 7 in the thousands place, numbers with 5, numbers with 3, and numbers with 1). Each of these groups contains one quarter of the numbers in the set. $(24/4 = 6)$. Alternatively, one can compute the number of ways to arrange 1, 3, and 7 into the hundreds, tens, and unit places after placing a 5 in the thousands place. There are $3 \times 2 \times 1 = 6$ such ways.

   (b) 5137.

   (c) 12. The only numbers smaller than 5137 in S are those with a 1 or a 3 in the thousands place. We know that both of these groups contain six items. $(6 + 6 = 12)$.

7. (a) 120 There are $5 \times 4 \times 3 \times 2 \times 1$ such numbers.

   (b) 31759. The 120 numbers can be divided into five groups with 24 numbers in each group based on the digit in the ten thousands place. The 24 smallest numbers begin with 1, the next 24 smallest begin with 3. Therefore 27th smallest will be the $3rd$-smallest beginning with 3 (since $27 - 24 = 3$). Since there are only 3 numbers to enumerate, one could list legal numbers (those using the right digits) in increasing order. However, we could also continue the process in the same manner on a smaller problem. We must find the 3rd smallest number that uses the digits 1, 5, 7, and 9 exactly once. The thousands digit partitions these numbers into four sets of six numbers as described in problem 6(a). Since $3 < 6$, our number is in the smallest group, the numbers starting with 1. We must then find the $3rd$ smallest number that uses the digits 5, 7, and 9 exactly once. There are three groups of 2 numbers depending upon the value of the hundreds digit. The 3rd-largest number is in the second group (numbers starting with 7). There are two smaller numbers in the first group, so we would like the smallest number starting with 7. This is 759. Therefore, returning to the original problem, the 27th largest is 31759.

8. 77. The five-term sequence starting with 77 is $77, 49, 36, 18, 8$. Because it's easy to compute the sequence for any 2-digit starting term, we can check all 90 possible starting terms. However, there are ways to reduce the number of starting terms we must check. Because the terms tend to get small quickly, we expect a 5-term sequence will start with a larger 2-digit number (we're more likley to have a long sequence starting with 99 than starting with 11). Therefore, it's best to start the search at 99 and move down rather than starting at 11 and moving upward. Also, swapping the digits doesn't effect the sequence (the product of the digits will not change). Therefore, the sequence computed for 72, for example, also gives us the sequence for 27.

9. (a) 26. There are 8 cousins with the 3 replaced (0 cannot be used), 9 cousins with the 6 replaced, and 9 cousins with the 4 replaced. This gives us $8 + 9 + 9 = 26$ cousins.

   (b) $864, 324, 360, 369$. To find a cousin of the form $A64$, we must find A such that $A+6+4$ is divisible by 9, or, since we can subtract multiples of 9 without effecting this divisibility, we require $A + 1$ be divisible by 9. There is only one such digit, namely $A = 8$. Similar procedures will find the other cousins where the second or third digits are changed.

10. and 11.

   (a) 4608. By rearranging the equation that defines V, we get $h + f + d + b = g + e + c + a$. In other words, from right to left, the sum of the digits in the odd-numbered slots (1st (units), 3rd (hundreds), 5th (ten thousands), and 7th (millions) slots) must equal the sum of the digits in the even-numbered slots (2nd (tens), 4th (thousands), 6th (hundred thousands), and 8th (ten millions) slots). Since $2 + 3 + 4 + 5 + 6 + 7 + 8 + 9 = 44$, each sum must equal 22. In particular, we must partition the digits $2, 3, 4, 5, 6, 7, 8, 9$ into two batches of four digits such that each batch sums to 22. Two such batches are called *complementary*.

   There are four pairs of complementary batches. We will now determine these batches and argue there are no more. Suppose that 9 and 8 are in the same batch. Since $9 + 8 = 17$, the remaining two members of the batch must sum to $22 - 17 = 5$. There is only one way to choose two digits from $2, 3, 4, 5, 6, 7$ that sum to 5, namely 2 and 3. Thus one pair of complementary batches is $\{2, 3, 8, 9\}, \{4, 5, 6, 7\}$. Any remaining complementary pairs must separate 8 and 9. By a similar argument, if we combine 9 and 7 in a batch, there is only one way to choose two digits from $2, 3, 4, 5$ that sum to $22 - 9 - 7 = 6$, namely 2 and 4. So another pair of complementary batches is {2,4,7,9}, {3,5,6,8}. Any remaining complementary batches must have 8 and 7 together. The remaining two digits in that batch must sum to $22 - 8 - 7 = 7$. There are only two ways to choose two digits from $2, 3, 4, 5, 6$ that sum to 7, namely $\{2, 5\}$ and $\{3, 4\}$. Thus the last two pairs are complementary batches are $\{2, 5, 7, 8\}$ with $\{3, 4, 6, 9\}$ and $\{3, 4, 7, 8\}$ with $\{2, 5, 6, 9\}$.

We now count the number of integers in both W and V. We may pick any of the four pairs of complementary batches. We can then assign either batch to the odd slots of our target integer, with the complementary batch assigned to the even slots. Thus there are 8 ways to choose the digits assigned to the odd/even slots. Once we have chosen which digits will be in the odd slots, there are are $4 \times 3 \times 2 \times 1 = 24$ ways to order these digits within the odd slots. There are also 24 ways to arrange the digits in the even slots. Therefore there are $8 \times 24 \times 24 = 4608$ integers in both V and W.

(b) 98734625. There are 4608 numbers in both V and W, so the 4600th term when the numbers are written in increasing order is the same as the 9th term when the numbers are written in decreasing order. By examining our batches of four digits, we see that the batch $\{9, 7, 4, 2\}$ and its compliment $\{8, 6, 5, 3\}$ will give us the largest numbers contained in both V and W. By assigning $\{9, 7, 4, 2\}$ to the even slots, we can set the first three digits to 987, the largest possible for W. The largest numbers will start with 9876. There are four such numbers because there are two ways to assign the remaining digits $(5, 3)$ to the first and third slots and two (independent) ways to place 2 and 4 as the second and fourth slots. The next four largest numbers are formed by replacing the 6 in the ten thousand place (5th digit from the right) with the next largest digit which is 5 (for example, 98754623). Finally, the next largest numbers are formed by replacing this 5 with the final choice of 3. We would like the largest such number obeying the odd/even slot assignments. This is 98734625.

12. $QP = 7, PT = 51$. We compute QP using the Pythagorean Theorem on triangle QPS. We have $(QP)^2 + (PS)^2 = (QS)^2$, so $QP = \sqrt{(25)^2 - (24)^2} = 7$. Triangles SRT and SPQ are both right triangles. They share the angle PSQ. Since triangles SRT and SPQ have two congruent angles, they are similar. Therefore, the ratios of their corresponding sides will be the same. The length SR is $47 + 25 = 72$. The ratio of SP to SR is the same as the ratio of QS to TS. Since $24 \times 3 = 72$, this ratio is 3. Thus $TS = 3 \times 25 = 75$ and $PT = 75 - 24 = 51$.

13. \$429,648. Let b represent the bonus and t represent the tax. Also, let $h = \$500,000$ and $n = h - t - b$ equal the desired net profit. We have the simultaneous equations

$$\begin{aligned} t &= .05(h-b) \\ 20t &= h-b \\ b+20t &= h. \end{aligned} \qquad\qquad \begin{aligned} b &= .10(h-t) \\ 10b &= h-t \\ 10b+t &= h. \end{aligned}$$

Equating the two expressions for h, we have

$$b + 20t = 10b + t$$

$$19t \quad = \quad 9b.$$

Also,

$$h = 10b + t = 10b + \frac{9}{19}b = \frac{199}{19}b = \frac{199}{19} \cdot \frac{19}{9}t = \frac{199}{9}t.$$

Finally,

$$n = h - t - b = h - \frac{9}{199}h - \frac{19}{199}h = \frac{171}{199}h = \frac{171 \times 500,000}{199} = \frac{85,500,000}{199} = \$429,648.$$

Note that the only rounding is in the last step. If you solve directly for b or t, you may introduce rounding error into both values and not determine h to the nearest dollar.